S-packing chromatic vertex-critical graphs

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Abstract

For a non-decreasing sequence of positive integers $S = (s_1, s_2, \ldots)$, the *S-packing chromatic number* $\chi_S(G)$ of G is the smallest integer k such that the vertex set of G can be partitioned into sets X_i, $i \in [k]$, where vertices in X_i are pairwise at distance greater than s_i. In this paper we introduce S-packing chromatic vertex-critical graphs, χ_S-critical for short, as the graphs in which $\chi_S(G-u) < \chi_S(G)$ for every $u \in V(G)$. This extends the earlier concept of the packing chromatic vertex-critical graphs. We show that if G is χ_S-critical, then the set $\{\chi_S(G) - \chi_S(G-u);\, u \in V(G)\}$ can be almost arbitrary. If G is χ_S-critical and $\chi_S(G) = k$ ($k \in \mathbb{N}$), then G is called k-χ_S-critical. We characterize 3-χ_S-critical graphs and partially characterize 4-χ_S-critical graphs when $s_1 > 1$. We also deal with k-χ_S-criticality of trees and caterpillars.

Keywords: packing coloring; S-packing coloring; S-packing vertex-critical graph

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1 Introduction

The *packing chromatic number* $\chi_\rho(G)$ of a graph G is the smaller integer k for which there exists a mapping $c : V(G) \to [k] = \{1, \ldots, k\}$, such that if $c(u) = c(v) = \ell$ for $u \neq v$, then $d_G(u, v) > \ell$. (Here and later $d_G(u, v)$ denotes the shortest-path distance between u and v in G.) Such a map c is called a *packing k-coloring*. This concept was introduced in [12], named with the present names in [4], and extensively studied afterwards. See [1, 2, 19, 5, 9, 17, 20], references therein, as well as [6] for a variant of a facial packing coloring.

A far reaching generalization of the packing chromatic number, formally introduced by Goddard and Xu in [13], but being implicitly present already in [12], is the following. Let $S = (s_1, s_2, \ldots)$ be a non-decreasing sequence of positive integers. An *S-packing k-coloring* of a graph G is a mapping $c : V(G) \to [k]$, such that if $c(u) = c(v) = \ell$ for $u \neq v$, then $d_G(u, v) > s_\ell$. The *S-packing chromatic number* $\chi_S(G)$ of G is the smallest integer k such that G admits an S-packing k-coloring. Note that if $S = (1, 1, 1, \ldots)$, then we are talking about the standard proper vertex coloring and if $S = (1, 2, 3, \ldots)$, then we deal with the packing coloring. For investigations of S-packing colorings see [7, 8, 10, 11, 14, 18].

Now, in [16] the *packing chromatic vertex-critical graphs* were introduced as the graphs G for which $\chi_\rho(G - u) < \chi_\rho(G)$ holds for every $u \in V(G)$. In this paper we are interested if (and how) the results from [16] extend from packing colorings to S-packing colorings. For this sake we say that G is *S-packing chromatic vertex-critical* (χ_S-critical for short) if $\chi_S(G - u) < \chi_S(G)$ holds for every $u \in V(G)$. And, when $\chi_S(G) = k$, we say that G is k-χ_S-critical.

We proceed as follows. In Section 2, we list some known results and prove two statements which will be used in the rest of the paper. In the subsequent section we consider the effect of vertex removal on the S-packing chromatic number and prove two realization theorems. Setting $\Delta_{\chi_S}(G) = \{\chi_S(G) - \chi_S(G - u) :\ u \in V(G)\}$, the first of these results asserts that if $S = (1^\ell, 2^\infty)$, $\ell \geq 1$, and $A = \{1, a_1, \ldots, a_k\}$, $k \geq 1$, is a set of positive integers, then there exists a χ_S-critical graph G such that $\Delta_{\chi_S}(G) = A$. In Section 4, we give a complete characterization of 3-χ_S-critical graphs while in Section 5 we partially characterize 4-χ_S-critical graphs for packing sequences with $s_1 > 1$. Finally, in Section 6 we show that a k-χ_S-critical tree exists for any positive integer k, investigate k-χ_S-criticality of caterpillars for sequences $S = (1, s_2^{k-1})$, and give some examples of such critical cattepillars.

2 Preliminaries

The order of a graph G will be denoted with $n(G)$. A graph consisting of a triangle and an edge with one end vertex on the triangle will be denoted by Z_1 (see also the top-right graph in Fig. 2).

We will be interested in non-decreasing finite or infinite sequences $S = (s_1, s_2, \ldots)$ of positive integers, but exclude the constant sequence $(1, 1, \ldots)$ because it leads to the chromatic number for which critical graphs are already well-studied, cf. [15]. For any other sequence S we will say that S is a *packing sequences*. If in a packing sequence a term i is repeated ℓ times, we will abbreviate the corresponding subsequence by i^ℓ. For instance, if $S = (1, \ldots, 1, s_{\ell+1}, \ldots)$, that is, if S starts with ℓ terms equal to 1, then we will write $S = (1^\ell, s_{\ell+1}, \ldots)$. Moreover, we will use the same convention for infinite constant subsequences. For example, $(1^\ell, 2, 2, \ldots)$ will be abbreviated $(1^\ell, 2^\infty)$.

Let G be a graph and $k \geq 1$. A set of vertices $A \subseteq V(G)$ is a *k-independent set* of G if A can be partitioned into k independent sets. The cardinality of a largest k-independent set of G is denoted by $\alpha_k(G)$. Note that $\alpha_k(G)$ can equivalently be defined as the maximum number of vertices of a graph G that can be properly colored using k colors. We now recall a series of results from [13] that will be used later.

Lemma 2.1 ([13, Observation 2]) *Let S be a packing sequence. If H is a subgraph of G, then $\chi_S(H) \leq \chi_S(G)$.*

Proposition 2.2 ([13, Proposition 4]) *Let $S = (s_1, s_2, \ldots)$ be a packing sequence and let G be a nonempty connected graph.*

1. *If $s_1 = s_2 = 1$, then $\chi_S(G) = 2$ if and only if G is bipartite.*

2. *If $s_1 = 1$ and $s_2 > 1$, then $\chi_S(G) = 2$ if and only if G is a star.*

3. *If $s_1 \geq 2$, then $\chi_S(G) = 2$ if and only if $G \simeq K_2$.*

Proposition 2.3 ([13, Proposition 6]) *Let $S = (1^\ell, s_{\ell+1}, \ldots)$, where $\ell \geq 1$ and $s_{\ell+1} \geq 2$, and let G be a graph with $\mathrm{diam}(G) = 2$. Then $\chi_S(G) = n(G) - \alpha_\ell(G) + \min\{\ell, \chi(G)\}$.*

Proposition 2.4 ([13, Proposition 20]) *Let G be a connected graph and $S = (2, 2, 2)$. Then G has a χ_S-packing coloring if and only if G is a path of any length or a cycle of lenght a multiple of 3.*

Proposition 2.5 ([13, Corollary 21]) *Let G be a connected graph and $S = (s_1, s_2, s_3)$, where $s_1 = 2$ and $s_3 \geq 3$, be a packing sequence. If G has a χ_S-coloring, then $n(G) \leq 5$.*

We conclude the preliminaries with two simple observations.

Lemma 2.6 *If S is a packing sequence and G is a χ_S-critical graph, then G is connected.*

Proof. Since $\chi_S(G) = \max_i\{\chi_S(G_i)\}$, where G_i are components of G, it follows that G must have only one component provided G is χ_S-critical. $\square$

Lemma 2.7 *Let S be a packing sequence. If u is a leaf of a graph G, then $\chi_S(G)-1 \leq \chi_S(G-u) \leq \chi_S(G)$.*

Proof. By Lemma 2.1, $\chi_S(G-u) \leq \chi_S(G)$. Suppose that $\chi_S(G-u) = k$. Then using an optimal S-packing coloring of $G-u$, and using color $k+1$ for the vertex u in G, we obtain that $\chi_S(G) \leq k+1$. Thus, $\chi_S(G) \leq \chi_S(G-u)+1$. $\square$

3 Vertex-deleted subgraphs of χ_S-critical graphs

In [16, Theorem 3.1] it was shown that if G is a χ_ρ-critical graph, then the set of differences $\Delta_{\chi_\rho}(G) = \{\chi_\rho(G) - \chi_\rho(G-u) :\ u \in V(G)\}$ can be almost arbitrary. Hence the condition $\chi_\rho(G-u) < \chi_\rho(G)$ for G to be χ_ρ-critical cannot be replaced with $\chi_\rho(G-u) = \chi_\rho(G) - 1$. We now show with a bit more involved construction than the one from [16] that the same phenomenon (actually even more general) holds for all sequences of the form $(1^\ell, 2^\infty)$. More precisely, setting

$$\Delta_{\chi_S}(G) = \{\chi_S(G) - \chi_S(G-u) :\ u \in V(G)\}$$

we have the following result.

Theorem 3.1 *Let $S = (1^\ell, 2^\infty)$, $\ell \geq 1$, and let $A = \{1, a_1, \ldots, a_k\}$, $k \geq 1$, be a set of positive integers. Then there exists a χ_S-critical graph G such that $\Delta_{\chi_S}(G) = A$.*

Proof. We may assume without loss of generality that $2 \leq a_1 < \cdots < a_k$.

Take a cycle of length $2k+1$ on vertices $x_1, \ldots, x_{2k+1}$ (in the natural order) and additional disjoint cliques Q_i, $i \in [2k+1]$, where $n(Q_i) = a_{\lceil i/2 \rceil} + \ell - 1$ for $i \in [2k]$ and $n(Q_{2k+1}) = a_k + \ell - 1$. Then for every $i \in [2k+1]$ and for every vertex x of Q_i, make x adjacent to each of the vertices x_i, x_{i+1}, and $x_{i+3}, x_{i+5}, \ldots, x_{i-2}$, where indices are taken modulo $2k+1$. Denote the constructed graph with $G(\ell; a_1, \ldots, a_k)$, see Fig. 1 for the graph $G(2; 2, 4)$.

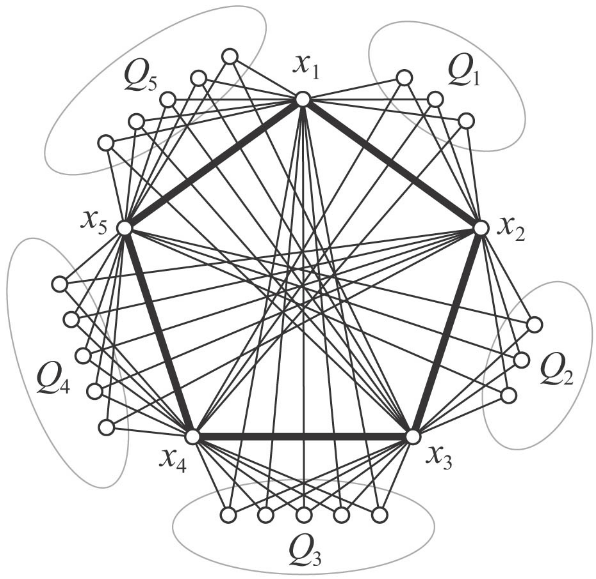

Figure 1: The graph $G(2; 2, 4)$

Note that the vertices of the clique Q_i together with the vertices x_i and x_{i+1} form a clique of order $a_{\lceil i/2 \rceil} + \ell + 1$. Denote this latter clique with Q'_i. To simplify the notation we set $G = G(\ell; a_1, \ldots, a_k)$ for the rest of the proof.

We claim first that $\mathrm{diam}(G) = 2$. For each $u_i \in Q_i$ and each $u_j \in Q_j$, $d_G(u_i, u_j) \leq 2$ since some of x_j, x_{j+1} is adjacent to both u_i and u_j by the definition of G. Analogously, for any x_i and any $u \in Q_j$, either $x_i u \in E(G)$ or $x_{i+1} u \in E(G)$ and clearly $x_i x_{i+1} \in E(G)$, implying that $d_G(x_i, u) \leq 2$. Finally, for any x_i and x_j, either $x_i x_j \in E(G)$ or x_i is adjacent to each vertex of Q_{j-1} or Q_j, implying that $d_G(x_i, x_j) \leq 2$ since $x_j u_j \in E(G)$ for every $u_j \in Q_{j-1} \cup Q_j$.

We have thus shown that $\mathrm{diam}(G) = 2$. Note that each clique Q'_i has order greater than ℓ, hence $\chi(G) \geq \omega(G) > \ell$. Then Proposition 2.3 implies that $\chi_S(G) = n(G) - \alpha_\ell(G) + \ell$. Moreover, $\alpha(G) = 2k+1$ and selecting ℓ vertices from each of the cliques Q_i we find an ℓ-independent set of order $\ell(2k+1)$. So $\alpha_\ell(G) = \ell(2k+1)$ and consequently

$$\begin{aligned}\chi_S(G) = n(G) - 2k\ell &= (2k+1) + 2\sum_{i=1}^{k}(a_i + \ell - 1) + (a_k + \ell - 1) - 2k\ell \\ &= (2k+1) + 2\sum_{i=1}^{k} a_i + 2k\ell - 2k + (a_k + \ell - 1) - 2k\ell = 2\sum_{i=1}^{k} a_i + a_k + \ell\,.\end{aligned}$$

Let $i \in [2k+1]$ and let x be an arbitrary vertex of Q_i. Having in mind that $n(Q_i) \geq \ell + 1$ we can repeat the above argument on the graph $G - x$ to get $\chi_S(G - x) =$

$2\sum_{i=1}^{k} a_i + a_{k+1} + \ell - 1$. Hence $\chi_S(G) - \chi_S(G - x) = 1$.

Consider now the graph $G - x_{2i}$, where $i \in [k]$. Note that in G, $n(Q'_{2i-1}) = n(Q'_{2i}) = a_i + \ell + 1$ and that x_{2i} is the unique common vertex of Q'_{2i-1} and Q'_{2i}. If $u \in V(Q_{2i-1})$ and $v \in V(Q_{2i})$, then in G, the vertex x_{2i} is the unique common neighbor of u and v. It follows that $d_{G-x_{2i}}(u, v) = 3$. On the other hand, for any other pair of vertices u' and v' of $G - x_{2i}$ we have $d_{G-x_{2i}}(u', v') = d_G(u', v')$. In particular, $d_{G-x_{2i}}(x_{2i-1}, x_{2i+1}) = 2$.

Since $\alpha_\ell(G - x_{2i}) = \ell(2k+1)$, we can select ℓ vertices from each of the cliques Q_i to form the first ℓ color classes. In addition, in $V(Q'_{2i-1}) - \{x_{2i}\}$ and $V(Q'_{2i}) - \{x_{2i}\}$ we have $a_i - 1$ pairs of vertices that are pairwise at distance 3. We can respectively color these pairs with colors $\ell + 1, \ldots, \ell + a_i - 1$. Because of the distances, every other not yet colored vertex requires its private color. Hence, with respect to the above optimal coloring of G, we have saved $a_i - 1$ colors. Since clearly $n(G - x_{2i}) = n(G) - 1$, we thus have $\chi_S(G - x_{2i}) = 2\sum_{i=1}^{k} a_i + a_{k+1} + \ell - (a_i - 1) - 1$, which in turn implies that $\chi_S(G) - \chi_S(G - x_{2i}) = a_i$.

We have proved by now that $A \subseteq \Delta_{\chi_S}(G)$. Finally, since $a_i < a_{i+1}$, by arguments parallel to the above arguments for x_{2i} we deduce that $\chi_S(G) - \chi_S(G - x_{2i+1}) = a_i$. We conclude that $A = \Delta_{\chi_S}(G)$. □

We proceed with the case where a packing sequences contains an element which is at least 3. To prove that the set of differences $\Delta_{\chi_S}(G)$ can be almost arbitrary for this case, we can follow the same line of thought than in the proof of [16, Theorem 3.1] for packing colorings, with a few key differences in the proof. What follows is the S-packing coloring version of this theorem.

Theorem 3.2 *Let S be a packing sequence such that there exists $\ell \geq 1$ with $s_\ell \geq 3$, and let $A = \{1, a_1, \ldots, a_k\}$, $k \geq 1$, be a set of positive integers. If for every $i \in [k]$ we have* $\sum_{j=1, j\neq i}^{k} a_j \geq a_i - 1$, *then there exists a χ_S-critical graph G such that $\Delta_{\chi_S}(G) = A$.*

Proof. Let $S = (s_1, s_2, \ldots)$ be a packing sequence and $\ell \geq 1$ the smallest positive integer such that $s_\ell \geq 3$.

First suppose that $k \geq 2$, and let $V(K_k) = \{x_1, \ldots, x_k\}$. We denote by $G(\ell; a_1, \ldots, a_k)$ the graph obtained from K_k such that for every $i \in [k]$, a vertex of a complete graph X_i of order $a_i + \ell - 1$ is identified with x_i. Again, we simplify the notation by setting $G = G(\ell; a_1, \ldots, a_k)$ in the remainder of the proof. We first observe that

$$n(G) = \sum_{i=1}^{k} n(X_i) = \sum_{i=1}^{k} (a_i + \ell - 1) = \sum_{i=1}^{k} a_i + k(\ell - 1).$$

If c is a χ_S-coloring of G, then the vertices of X_i, $i \in [k]$, receive pairwise different colors. To be more precise, we have $|c^{-1}(j)| \leq k$ for any $j \leq \ell - 1$. Moreover, since $\mathrm{diam}(G) = 3$, $|c^{-1}(j)| \leq 1$ for any $j \geq \ell$. Since $a_i \geq 2$, and so $a_i + \ell - 1 \geq \ell + 1$, in each X_i colors $1, \ldots, \ell - 1$ can be used. Therefore,

$$\chi_S(G) = (\ell - 1) + (n(G) - k(\ell - 1)) = (\ell - 1) + \sum_{i=1}^{k} a_i. \tag{1}$$

Since $k \geq 2$, for at least one a_i we have $a_i \geq 3$. Without loss of generality we can assume that $a_1 \geq 3$. Let $u \in V(X_1)$ be an arbitrary vertex different from x_1. Then $G - u$ is isomorphic to $G(\ell; a_1 - 1, a_2, \ldots, a_k)$ (it is possible that $a_1 - 1 = a_i$ for some $i \geq 2$). By (1) we get

$$\chi_S(G - u) = (\ell - 1) + (a_1 - 1) + \sum_{i=2}^{k} a_i = \sum_{i=1}^{k} a_i + (\ell - 2) = \chi_S(G) - 1.$$

This shows that $1 \in \Delta_{\chi_S}(G)$.

Now we consider the vertex-deleted subgraph $G - x_i$, $i \in [k]$. Since x_i is a cut-vertex, and using (1), we have

$$\begin{aligned}
\chi_S(G - x_i) &= \max\{\chi_S(K_{a_i+\ell-2}), \chi_S(G(\ell; a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_k))\} \\
&= \max\left\{a_i + \ell - 2, (\ell - 1) + \sum_{j=1, j\neq i}^{k} a_j\right\} \\
&= (\ell - 1) + \sum_{j=1, j\neq i}^{k} a_j,
\end{aligned}$$

where the last inequality follows from the assumption $\displaystyle\sum_{j=1,j\neq i}^{k} a_j \geq a_i - 1$. It follows that

$$\chi_S(G) - \chi_S(G - x_i) = \left((\ell - 1) + \sum_{i=1}^{k} a_i\right) - \left((\ell - 1) + \sum_{j=1,j\neq i}^{k} a_j\right) = a_i,$$

and we get $a_i \in \Delta_{\chi_S}(G)$ for every $i \in [k]$.

Suppose now that $k = 1$, and $A = \{1, a\}$, where $a \geq 2$. In this case, let $\ell \geq 1$ be the smallest index with $s_\ell \geq 2$ (not 3 as in the previous case). Let G be the graph obtained from two disjoint copies of $K_{a+\ell-1}$ by identifying a vertex from one copy with a vertex from the other copy, and let x be the identified vertex. We have

$n(G) = 2(a+\ell-1)-1 = 2(a+\ell)-3$ and $\chi_S(G) = (\ell-1)+(n(G)-2(\ell-1)) = 2a+\ell-2$, since $\mathrm{diam}(G) = 2$. If u is a vertex of G different from x, then $\chi_S(G-u) = (\ell-1) + ((n(G)-1)-2(\ell-1)) = 2a+\ell-3$. Thus, $\chi_S(G)-\chi_S(G-u) = 1$. To end the proof, we notice that $\chi_S(G-x) = \chi_S(K_{a+\ell-2}) = a+\ell-2$, and hence $\chi_S(G)-\chi_S(G-x) = a$. $\square$

4 3-χ_S-critical graphs

If S is an arbitrary packing sequence, then it is clear that K_2 is the unique 2-χ_S-critical graph. In the following theorem we give a complete list of all 3-χ_S-critical graphs with respect to a given packing sequence S.

Theorem 4.1 *Let S be a packing sequence and let G be a graph.*

1. *If $S = (1, 1, \ldots)$, then G is 3-χ_S-critical if and only if $G \in \{C_{2k+1} : k \geq 1\}$.*
2. *If $S = (1, s_2, \ldots)$, $s_2 \geq 2$, then G is 3-χ_S-critical if and only if $G \in \{C_3, C_4, P_4\}$.*
3. *If $S = (s_1, s_2, \ldots)$, $s_1 \geq 2$, then G is 3-χ_S-critical if and only if $G \in \{C_3, P_3\}$.*

Proof. Let G be a 3-χ_S-critical graph. Then G is connected by Lemma 2.6. Clearly, $n(G) \geq 3$. If $n(G) = 3$, then G is either C_3 or P_3. Clearly, C_3 is 3-χ_S-critical for every packing sequence S, while P_3 is 3-χ_S-critical exactly for packing sequences $S = (s_1, s_2, \ldots)$, where $s_1 \geq 2$. For the rest of the proof we may thus assume that $n(G) \geq 4$.

Let $u \in V(G)$ be an arbitrary vertex of G. Since G is a 3-χ_S-critical graph, we have $\chi_S(G-u) = 2$ or $\chi_S(G-u) = 1$. The later case means that $G-u$ is a disjoint union of isolated vertices, and hence G would be 2-colorable for every packing sequence S. Henceforth $\chi_S(G-u) = 2$ holds. We now distinguish three cases with respect to the shape of S.

Case 1. $s_1 = s_2 = 1$.
In this case it is clear that $\chi(G-u) = \chi_S(G-u) = 2$, and hence $G-u$ is a disjoint union of connected bipartite graphs and isolated vertices. If u would be adjacent to vertices from at most one partition of each of the bipartite graphs, then G would be 2-colorable. Thus there exists at least one connected bipartite component of $G-u$, say G_1, such that u has neighbors in both bipartition sets of G_1 and such that the subgraph induced by $V(G_1) \cup \{u\}$ contains an odd cycle C_{2k+1}, $k \geq 1$. Then $V(G) = V(C_{2k+1})$, for otherwise removing a vertex not belonging to the cycle would yield a 3-colorable

graph. Moreover, $E(G) = E(C_{2k+1})$, for otherwise an additional edge would yield a shorter odd cycle, so G would not be 3-χ_S-critical.

Case 2. $s_1 = 1$, $s_2 \geq 2$.
In this case we can follow a similar line of thought than in the proof of [16, Proposition 4.1]. In view of Proposition 2.2, if u is a vertex of a 3-χ_S-critical graph G, then $G - u$ is a disjoint union of stars and isolated vertices. It is clear that $G - u$ must contain at least one star, say G_1, for otherwise G would itself be a star. If $G - u$ contains more than one star, then G contains P_5, and cannot be 3-χ_S-critical, because by removing an end-vertex of P_5, the obtained graph would contain a P_4 for which $\chi_S(P_4) = 3$ for every packing sequence S with $s_2 \geq 2$. Also, if $G - u$ has more than one isolated vertex, then removing one such vertex from G yields a graph with $\chi_S = 3$. Thus, $G - u$ contains one star an at most one isolated vertex.

First suppose that $G - u = G_1$. Since $n(G) \geq 4$, the star G_1 must have at least two leaves. If u is adjacent to the center of G_1, then since G itself is not a star, u is adjacent to at least one leaf of G_1. Removing one of the other leaves (say v) in G_1 gives a graph that contains C_3, and $\chi_S(G - v) = 3$, which is a contradiction. Therefore, u is not adjacent to the center of G_1, hence it is adjacent to at least one leaf of G_1. If G_1 contains at least three leaves, then G is not 3-χ_S-critical because then we can remove a leaf and the obtained graph contains P_4, for which, as already noticed, $\chi_S(P_4) = 3$ for every packing sequence S with $s_2 \geq 2$. Hence G_1 must have exactly two leaves. If u is adjacent to exactly one of them, we get P_4, and if it is adjacent to both of them, we get C_4. Both graphs are 3-χ_S-critical for any packing sequence S with $s_2 \geq 2$.

The other case to consider is when $G - u$ is a disjoint union of the star G_1 and an isolated vertex, say w. If G_1 has at least two leaves, then we deduce as in the subcase above that G is not 3-χ_S-critical. And if $G_1 = K_2$, then G is either P_4, which is 3-χ_S-critical for any packing sequence S with $s_2 \geq 2$, or Z_1 which is not 3-χ_S-critical.

Case 3. $s_1 \geq 2$.
In this case, Proposition 2.2 implies that if u is a vertex of a 3-χ_S-critical graph G, then $G - u$ is a disjoint union of copies of K_2 and isolated vertices. If $G - u$ has more than one K_2, then G contains P_5 and cannot be 3-χ_S-critical, since by removing an end-vertex of P_5 the obtained graph would still have a P_4 which is 3-χ_S-colorable since $s_1 \geq 2$. Also, if $G - u$ contains isolated vertices, then G contains a path P_4, and removing a vertex from this path yields a P_3 in $G - u$. Thus, $\chi_S(G - u) = 3$ for any packing sequence S with $s_1 \geq 2$, which is a contradiction. We conclude that $G - u = K_2$ must hold and therefore $G = C_3$ or P_3. Both remaining graphs are 3-χ_S-critical for every packing sequence with $s_1 \geq 2$. $\square$

Note that Theorem 4.1 implies that C_3 is the unique graph that is 3-χ_S-critical for every packing sequence S.

5 On 4-χ_S-critical graphs

In this section we deal with the 4-χ_S-critical graphs for packing sequences with $s_1 \geq 2$. All critical graphs from appear in Theorem 5.1 are depicted in Fig. 2.

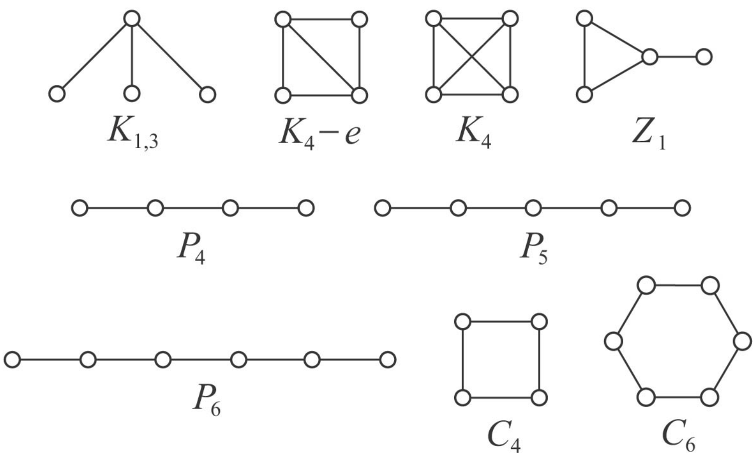

Figure 2: The 4-χ_S-critical graphs for packing sequences S with $s_1 \geq 2$

Theorem 5.1 *Let $S = (s_1, s_2, \ldots)$ be a packing sequence with $s_1 \geq 2$, and let G be a graph.*

1. *If $s_3 = 2$, then G is 4-χ_S-critical if and only if $G \in \{K_{1,3}, C_4, Z_1, K_4 - e, K_4\}$.*
2. *If $s_2 = 2$ and $s_3 \geq 3$, then G is 4-χ_S-critical if and only if $G \in \{K_{1,3}, C_4, Z_1, K_4 - e, K_4, P_6, C_6\}$.*
3. *If $s_1 = 2$ and $s_2 \geq 3$, then G is 4-χ_S-critical if and only if $G \in \{K_{1,3}, C_4, Z_1, K_4 - e, K_4, P_5\}$.*
4. *If $s_1 \geq 3$, then G is 4-χ_S-critical if and only if $G \in \{K_{1,3}, P_4, C_4, Z_1, K_4 - e, K_4\}$.*

Proof. Let G be a 4-χ_S-critical graph. Then G is connected by Lemma 2.6. Clearly, $n(G) \geq 4$. Since $s_1 \geq 2$ we have $\Delta(G) \leq 3$, for otherwise G is not 4-χ_S-colorable ($\chi_S(G) \geq \Delta(G) + 1$).

Claim 1 *Let G be a 4-χ_S-critical graph different from $K_{1,3}$ and let $u \in V(G)$ be such that $\chi_S(G-u) = 3$. If $G-u$ is disconnected, then $d_G(x) \le 2$ for any $x \in V(G-u)$.*

Proof. Suppose that $G-u$ consists of at least two components. Let to the contrary G contain a vertex $x \neq u$ of degree 3, and let G_1 denote the component of $G-u$ containing x. Then, deleting any vertex y of G not belonging to G_1 yields a graph with $d_{G-y}(x) = 3$, implying that $G-y$ is not 3-χ_S-colorable, a contradiction. □

Case 1. $s_1 = 2$.
First suppose that $\chi_S(G-u) = 1$ for each $u \in V(G)$. Then, clearly, $G-u$ consists of isolated vertices. Since $n(G) \ge 4$ and $d_G(u) \le 3$, we get $K_{1,3}$. But deleting any leaf of it we get a graph which is not 1-χ_S-colorable, which is a contradiction.

Now suppose that for each vertex u' of G, $\chi_S(G-u') \le 2$ and there exists a vertex $u \in V(G)$ such that $\chi_S(G-u) = 2$. By Proposition 2.2, $G-u$ is a disjoint union of at least one copy of K_2 and some (possibly zero) isolated vertices. Clearly $d_G(u) \le 3$. Since $n(G) \ge 4$, $G-u$ is disconnected and hence $d_G(u) \ge 2$. If $d_G(u) = 2$, then $G-u$ consists of two components and u has a neighbor in each of them, implying that $G \simeq P_k$, where $k \in \{4,5\}$. If $k = 4$, then $\chi_S(P_k) = 3$ whenever $s_1 = 2$, which is a contradiction; if $k = 5$, then $\chi_S(P_k) = 4$ when $s_2 \ge 3$, otherwise $\chi_S(P_k) = 3$. Thus the only critical graph for $s_2 \ge 3$ is P_5.

Now assume that $d_G(u) = 3$. If some vertex v of $G-u$ is not adjacent to u in G, then $d_{G-v}(u) = 3$, implying that $\chi_S(G-v) \ge 4$ and hence G is not 4-χ_S-critical. If $G-u$ consists of three components, then each of these components must be K_1, implying that $\chi_S(G-u) = 1$, which is a contradiction. Thus $G-u$ has two components and exactly one of them has two vertices. Then we get $G \simeq Z_1$, but deleting the leaf of Z_1 we get C_3 which is not 2-χ_S-colorable, which is again a contradiction.

Finally suppose that G contains a vertex u such that $\chi_S(G-u) = 3$. Clearly, $n(G-u) \ge 3$ and $d_G(u) \le 3$.

Assume that $d_G(u) = 3$. If some vertex v of $G-u$ is not adjacent to u in G, then $\Delta(G-v) = 3$, implying that $\chi_S(G-v) \ge 4$ and hence G is not 4-χ_S-critical. Thus $n(G-u) = 3$. If $G-u$ is disconnected, then each component of $G-u$ has at most 2 vertices and hence $\chi_S(G-u) \le 2$, which is a contradiction. Thus $G-u$ is connected, $G-u \simeq G' \in \{P_3, C_3\}$, implying that $G \in \{K_4 - e, K_4\}$.

Now we assume that $d_G(u) \le 2$ and consider the following possibilities.

Subcase 1.1 $s_3 = 2$ (and also $s_1 = s_2 = 2$).
By Proposition 2.4, $G-u$ consists of a disjoint union of K_1, K_2, some paths of arbitrary lengths, and of cycles of lengths divisible by 3.

Assume that $d_G(u) = 1$. Since G is connected and $n(G) \geq 4$, we infer that $G - u \simeq G' \in \{P_k : \ k \geq 3\} \cup \{C_{3k} : \ k \geq 1\}$. Connecting u to an end-vertex of any path we get a path which is still 3-χ_S-colorable, which is a contradiction. Connecting u to a vertex of degree 2 of any path we get $G \simeq K_{1,3}$, or a graph which is not 4-χ_S-critical, since for any leaf $v \in V(G)$, such that v is adjacent to a vertex of degree 2 in G, we get $\Delta(G - v) = 3$ implying that $\chi_S(G - v) \geq 4$. Analogously, connecting u to a vertex of a C_k, $k > 3$, we get a graph which is not 4-χ_S-critical. Connecting u with one vertex of C_3 we get $G \simeq Z_1$.

Assume that $d_G(u) = 2$ and recall that $n(G - u) \geq 3$. If $G - u$ is connected, then, for $G - u \simeq P_3$ we get $G \simeq C_4$ or $G \simeq Z_1$, for $G - u \simeq C_3$ we get $G \simeq K_4 - e$, while in any other case we get a graph which is not 4-χ_S-critical or is 3-χ_S-colorable (a path). Thus let $G - u$ be disconnected and consisting of two components G_1 and G_2. By Claim 1, u is adjacent to vertices of degree 1 only. Thus G_1 and G_2 are both paths and u is adjacent to one end-vertex of G_1 and one endvertex of G_2. Then G is a path, hence it is 3-χ_S-colorable, which is a contradiction.

Subcase 1.2. $s_2 = 2$ and $s_3 \geq 3$.
By Proposition 2.5, $n(G - u) \leq 5$. Clearly $\Delta(G - u) \leq 2$ since $\chi_S(G - u) = 3$ and $\chi_S(G) \geq \Delta(G) + 1$. Since none of C_4 and C_5 is 3-χ_S-colorable, $G - u$ is a disjoint union of some copies of K_1, K_2, P_3, C_3, P_4, and/or P_5.

Assume that $d_G(u) = 1$. Since G is connected and $n(G) \geq 4$, the graph $G - u$ must be one of P_3, C_3, P_4, and P_5. If $G - u \simeq C_3$, then $G \simeq Z_1$. If u is adjacent to an end-vertex of P_3, P_4, or P_5, then we either get a 3-χ_S-colorable graph or P_6, hence $G \simeq P_6$. If u is adjacent to the central vertex of P_3, then $G \simeq K_{1,3}$. If u is adjacent to some vertex of P_4 or P_5 of degree 2, then we get a graph which is not 4-χ_S-critical since $\chi_S(G - v) = 4$ for any $v \in V(G)$ with $d_G(v) = 1$, and v is adjacent to a vertex of degree 2 in G.

Assume that $d_G(u) = 2$. If $G - u$ is not connected, then u has a neighbor in two distinct components of $G - u$, implying that $\Delta(G) = 2$ by Claim 1. Thus we get $G \simeq P_6$ since P_4 and P_5 are both 3-χ_S-colorable. If $G - u$ is connected, then $G - u \in \{P_3, C_3, P_4, P_5\}$ since $n(G) \geq 4$. Then, for $G - u \simeq P_3$ we get $G \in \{C_4, Z_1\}$, for $G - u \simeq C_3$ we get $G \in \{K_4 - e, K_4\}$, and for $G - u \simeq P_5$ we get $G \simeq C_6$; in any other case G is not 4-χ_S-critical.

Subcase 1.3. $s_2 \geq 3$.
Since $s_2 \geq 3$, $s_3 \geq 3$ as well. Hence, by Proposition 2.5, $n(G - u) \leq 5$. Clearly, $\Delta(G - u) \leq 2$. Since none of P_5, C_4, C_5 is 3-χ_S-colorable, $G - u$ is a disjoint union of some copies of K_1, K_2, P_3, C_3 and/or P_4.

Assume that $d_G(u) = 1$. Since $n(G) \geq 4$, $G - u \simeq G' \in \{P_3, C_3, P_4\}$. If $G - u \simeq P_3$, then connecting u with the central vertex of P_3 we get $G \simeq K_{1,3}$, otherwise connecting u with an end-vertex of P_3 we get a 3-χ_S-colorable graph P_4, which is a contradiction. If $G - u \simeq P_4$, then connecting u with an end-vertex of P_4 we get $G \simeq P_5$, otherwise connecting u with some vertex of degree 2 we get a graph which is not 4-χ_S-critical. And, if $G - u \simeq C_3$, we get $G \simeq Z_1$.

Assume that $d_G(u) = 2$. If $G - u$ is not connected, then u has a neighbor in two distinct components of $G - u$, implying that $\Delta(G) = 2$ by Claim 1. Thus we get $G \simeq P_5$, since P_k is not 4-χ_S-critical for any $k \geq 6$ and P_4 is 3-χ_S-colorable. If $G - u$ is connected, then $G - u \in \{P_3, C_3, P_4\}$ since $n(G) \geq 4$. Then, for $G - u \simeq P_3$ we get $G \in \{C_4, Z_1\}$ and for $G - u \simeq C_3$ we get $G \simeq K_4 - e$; in any other case G is not 4-χ_S-critical.

Case 2. $s_1 \geq 3$.
If $\Delta(G) = 3$, then $n(G) = 4$, implying that $G \in \{K_{1,3}, Z_1, K_4 - e, K_4\}$, for otherwise we get a graph which is not 4-χ_S-colorable. If $\Delta(G) = 2$, then since G is connected, $G \simeq P_k$ or C_k, $k \geq 4$. Clearly P_k is not 4-χ_S-critical for any $k \geq 5$, hence we get $G \simeq P_4$. For cycles, C_k is 4-χ_S-colorable if and only if $k = 4$, or $s_4 = 3$ and k is divisible by 4. And, when $k > 4$, C_k is not 4-χ_S-critical since $\chi_S(C_k - u) = 4$ for any $u \in V(C_k)$ since $C_k - u$ contains a P_4 for which $\chi_S(P_4) = 4$. Thus $G \simeq C_4$. Note that the graphs $K_{1,3}$, P_4 and C_4 are all 4-χ_S-critical.

Finally note that each of the graphs from the set $\{K_{1,3}, C_4, Z_1, K_4 - e, K_4\}$ are 4-χ_S-critical for every packing sequence S with $s_1 \geq 2$, each of the graphs P_6 and C_6 are 4-χ_S-critical for $s_2 = 2$ and $s_3 \geq 3$, the graph P_5 is 4-χ_S-critical for $s_1 = 2$ and $s_2 \geq 3$, and the graph P_4 is 4-χ_S-critical for $s_1 \geq 3$. □

6 χ_S-critical trees

Let S be a packing sequence with $s_1 = s_2 = 1$. Then every bipartite graph, in particular every tree, admits a 2-χ_S-coloring. It follows that K_2 is the only χ_S-critical graph for such a packing sequence. On the other hand, if $s_2 \geq 2$, then the situation is more interesting already on trees as the next result which extends [16, Proposition 5.1] asserts.

Proposition 6.1 *If $k \geq 2$ and S is a packing sequence with $s_2 \geq 2$, then there exists a k-χ_S-critical tree.*

Proof. Suppose first that $s_1 \geq 2$ (and, of course, $s_2 \geq s_1$). Then $K_{1,k-1}$ is a required k-χ_S-critical tree.

Assume in the rest that $s_1 = 1$ and (and $s_2 \geq 2$). Let T_k be the tree obtained from $K_{1,k-1}$ with the central vertex u and leaves $w_1, \dots, w_{k-1}$, by attaching $k-2$ leaves to each of the vertices w_i, $i \in [k-1]$. In particular, $T_2 = K_2$ and $T_3 = P_5$.

We claim that $\chi_S(T_k) = k$. Let c be an arbitrary χ_S-coloring of T_k. If $c(w_i) = 1$ holds for some $i \in [k-1]$, then the $k-1$ neighbors of w_i must receive pairwise different colors, hence c uses at least k colors. On the other hand, if $c(w_i) \neq 1$ for each $i \in [k-1]$, then c uses $k-1$ colors on the vertices w_i and hence at least k colors all together. This shows that $\chi_S(T_k) \geq k$. On the other hand, setting $c(w_i) = i+1$, $i \in [k-1]$, and coloring every other vertex with color 1 yields $\chi_S(T_k) \leq k$. This proves the claim.

We have thus seen that $\chi_S(T_k) = k$. If T_k is k-χ_S-critical, then we are done. Otherwise, using Lemma 2.7, remove leaves of T_k one by one until a k-χ_S-critical tree is obtained. □

In [16], k-χ_S-critical caterpillars were investigated for $S = (1, 2, 3, \dots)$. Here we focus on existence of k-χ_S-critical caterpillars for some packing sequences with $s_1 = 1$. Recall that, for $s_2 = 1$, the only χ_S-critical graph is K_2, thus we consider $s_2 \geq 2$.

Proposition 6.2 *Let $k \geq 2$ be an integer and $S = (1, s_2^{k-1})$ a packing sequence with $s_2 \geq 2$. Then a k-χ_S-critical caterpillar exists if and only if $k \leq s_2 + 2$.*

Proof. Let T be a caterpillar. Since any path has a $(s_2^{s_2+1})$-packing coloring repeating the coloring pattern $2, 3, \dots, s_2 + 2$, we can color vertices of the spine of T with colors $2, 3, \dots s_2 + 2$ and then color all the leaves of T with color 1. Thus $\chi_S(T) \leq s_2 + 2$ for an arbitrary caterpillar T.

On the other hand, any path of length greater than s_2 has no $(s_2^{s_2})$-packing coloring. Then, considering any path P of length greater than s_2 and attaching at least $s_2 + 2$ leaves to each vertex of P we get a caterpillar T with $\chi_S(T) = s_2 + 2$. Iteratively applying Lemma 2.7 to the leaves of T we find an $(s_2 + 2)$-χ_S-critical caterpillar. Continuing deleting leaves of T in this manner we can also get a k-χ_S-critical caterpillar for any $k \leq s_2 + 1$. □

Now we construct an explicit k-χ_S-critical caterpillars for $S = (1, s_2^{k-1})$ and any $k \leq s_2 + 2$, $k \geq 2$.

Example 1 Let $S = (1, s_2^{k-1})$ be such that $k \leq s_2$. Let G_1 be a caterpillar consisting of a spine P of length $k-2$ and adding one leaf to each vertex of P (see Fig. 3 (a)). We show that G_1 is k-χ_S-critical. First, since each color different from 1 can be used for only one vertex of G_1 and color 1 can be used on at most $k-1$ vertices, we have $\chi_S(G_1) \geq k$. On the other hand, we can color vertices of P with colors $2, 3, \ldots, k$ and all leaves of G_1 with color 1, implying that $\chi_S(G_1) = k$.

Now we show that $\chi_S(G_1 - x) < k$ for any $x \in V(G_1)$. Deleting any leaf x of G_1 we get a graph in which we color all leaves of $G_1 - x$ and the neighbor of x in G_1 with color 1, and we color the remaining $k-2$ vertices of P with mutually distinct colors $2, 3, \ldots, k-2$, implying that $\chi_S(G_1 - x) < k$. If $x \in V(P)$, then $G_1 - x$ is disconnected and consisting of one isolated vertex and one or two caterpillars C_1, C_2. Clearly, each of C_1, C_2 has a spine of length smaller than $k-1$, implying that $\chi_S(C_i) < k$ for $i = 1, 2$. Therefore G_1 is k-χ_S-critical.

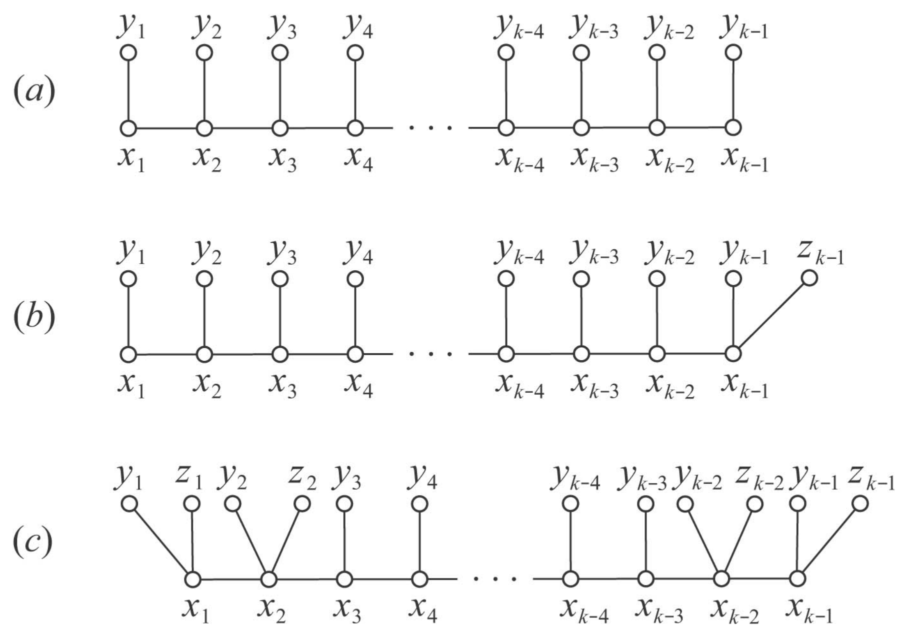

Figure 3: Caterpillars

Example 2 Let $S = (1, s_2^{k-1})$ be such that $k = s_2 + 1$. Let G_2 be a caterpillar consisting of a spine $P = x_1, x_2, \ldots, x_{k-1}$, adding one leaf y_i to x_i for each $i = 1, \ldots, k-1$ and adding one more leaf z_{k-1} to x_{k-1} (see Fig. 3 (b)). We show that G_2 is k-χ_S-critical. First, since color 1 can be used for an independent set of G_2, we have to color either x_i or all leaves adjacent to x_i with some of the colors $2, 3, \ldots, k$ for each $i = 1, \ldots, k-1$. And, since we can use only one such color twice (for y_1, and y_{k-1} or z_{k-1}), we have $\chi_S(G_2) \geq k$. On the other hand, coloring vertices of P with colors $2, 3, \ldots, k$ and coloring all leaves of G_2 with color 1, we get an k-χ_S-coloring of G_2, implying that $\chi_S(G_2) = k$.

Now we show that $\chi_S(G_2 - x) < k$ for any $x \in V(G_2)$. Deleting any leaf not adjacent to x_{k-1} we get a graph in which we color all leaves of $G_2 - x$ and the neighbor of x in G_2 with color 1, and we color the remaining $k-2$ vertices of P with mutually distinct colors $2, 3, \ldots, k-1$, implying that $\chi_S(G_2 - x) < k$. If x is adjacent to x_{k-1}, say, $x = z_{k-1}$, we color y_1 and y_{k-1} with color 2, all internal vertices of P (one by one) with colors $3, 4, \ldots, k-1$ and all the remaining vertices with color 1. Thus, in this case, $\chi_S(G - x) < k$. If $x \in V(P)$, then $G_2 - x$ is disconnected and it consists of one or two isolated vertices and one or two caterpillars C_1, C_2. Clearly, each of C_1, C_2 has a spine of length smaller than $k-1$, implying that $\chi_S(C_i) < k$ for $i = 1, 2$. Therefore G_1 is k-χ_S-critical.

Example 3 Let $S = (1, s_2^{k-1})$ be such that $k = s_2+2$. Let G_3 be a caterpillar consisting of a spine $P = x_1, x_2, \ldots, x_{k-1}$, adding one leaf y_i to x_i for each $i = 3, \ldots, k-3$, and adding two leaves y_i, z_i to x_i for each $i = 1, 2, k-2, k-1$ (see Fig. 3 (c)). We show that G_3 is k-χ_S-critical. First, since color 1 can be used for an independent set of G_3, we have to color either x_i or all leaves adjacent to x_i with some of the colors $2, 3, \ldots, k-1$ for each $i = 1, \ldots, k-1$. It is straightforward to check that we always need at least $k-1$ such colors, implying that $\chi_S(G_3) \geq k$. On the other hand, coloring vertices of P with colors $2, 3, \ldots, k$ and coloring all leaves of G_3 with color 1, we get an k-χ_S-coloring of G_3, implying that $\chi_S(G_3) = k$.

Now we show that $\chi_S(G_3 - x) < k$ for any $x \in V(G_3)$. Deleting any leaf $x = y_i$ for some $i = 3, 4, \ldots, k-3$, we get a graph in which we can color all leaves of $G_3 - x$ and the neighbor of x in G_3 with color 1, and we color the remaining $k-2$ vertices of P with mutually distinct colors $2, 3, \ldots, k-1$, implying that $\chi_S(G_3 - x) < k$. If, up to symmetry, $x = y_1$, then we can color x_1 and all leaves of $G_3 - x$ different from z_1 with color 1, z_1 and x_{k-1} with color 2, and the remaining $k-3$ vertices of P one by one with colors $3, \ldots, k-1$, implying that $\chi_S(G_3 - x) < k$. Analogously, if, up to symmetry, $x = y_2$, then we can color x_2, x_{k-1} and all leaves of $G_3 - x$ different from z_2, y_{k-1} and z_{k-1} with color 1, x_1 and y_{k-1} with color 2, z_2 and z_{k-1} with color 3, and the remaining $k-4$ vertices of P one by one with colors $4, 5, \ldots, k-1$, implying that $\chi_S(G_3 - x) < k$. If $x \in V(P)$, then $G_3 - x$ is disconnected and it consists of one or two isolated vertices and one or two caterpillars C_1, C_2. Clearly, each of C_1, C_2 has a spine of length smaller than $k-1$, implying that $\chi_S(C_i) < k$ for $i = 1, 2$. Therefore G_3 is k-χ_S-critical.

Concluding remarks

The criticallity studied in [16] for the packing chromatic number and the criticallity investigated in this paper for the S-packing chromatic number refer to vertex-deleted subgraphs. An equally legal criticallity concept is the one with respect to arbitrary subgraphs, equivalently with respect to edge-deleted subgraphs. The seminal study [3] on the latter concept for the packing chromatic number has been done independently and at about the same time as [16]. It would hence be natural to study also the edge-deleted critical graphs in the general context of S-packing colorings.

It was stated in [16] that it would be interesting to classify vertex-transitive, χ_ρ-critical graphs. Here we extend this claim by stating that it would also be of interest to classify vertex-transitive, χ_S-critical graphs for each of the packing sequence S.

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